

Combinatorics of  $\phi$ -deformed stuffle Hopf algebras<sup>1</sup>G rard H. E. Duchamp<sup>1 2</sup>Hoang Ngoc Minh<sup>2 3</sup>Christophe Tollu<sup>1 2</sup>Chi n B i<sup>4 2</sup>Hoang Nghia Nguyen<sup>1 2</sup><sup>1</sup>Universit  Paris 13, 99, avenue Jean-Baptiste Cl ment, 93430 Villetaneuse, France.<sup>2</sup>LIPN - UMR 7030, CNRS, 93430 Villetaneuse, France.<sup>3</sup>Universit  Lille II, 1, Place D liot, 59024 Lille, France.**Abstract**

In order to extend the Sch utzenberger's factorization to general perturbations, the combinatorial aspects of the Hopf algebra of the  $\phi$ -deformed stuffle product is developed systematically in a parallel way with those of the shuffle product and in emphasizing the Lie elements as studied by Ree. In particular, we will give an effective construction of pair of bases in duality.

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# 1 Introduction

Many algebras of functions [6] and many special sums [8, 9] are ruled out by shuffle products, their perturbations (adding a “superposition term” [7]) or deformations [17].

In order to better understand the mechanisms of this products, we wish here to examine, with full generality the products which are defined by a recursion of the type

$$au \star bv = a(u \star bv) + b(au \star v) + \phi(a, b) u \star v, \quad (1)$$

the empty word being the neutral of this new product.

We give a lot of classical combinatorial applications. In most cases, the law  $\phi$  is dual<sup>2</sup> and under some growth conditions the obtained algebra is an enveloping algebra.

The structure of the paper is the following `TODO` . . . .

In the second section, is a version of the CQMM without PBW. We are obliged to redo the CQMM theorem without supposing any basis because we aim at “varying the scalars” in forthcoming papers (germs of functions, arithmetic functions, etc.) and, in order to do this at ease, we must cope safely with cases where torsion may appear (and then, one cannot have any basis). See (counter) examples in the section.

## 2 ??

Let  $X$  be an totally ordered alphabet<sup>3</sup>. The free monoid and the set of Lyndon words, over  $X$ , are denoted respectively by  $X^*$  and  $\mathcal{Lyn}X$ . The neutral element of  $X^*$ , *i.e.* the empty word is denoted by  $1_{X^*}$ . Let  $\mathbb{Q}\langle X \rangle$  be equipped by the concatenation and the shuffle which is defined by

$$\begin{aligned} \forall w \in X^*, \quad w \sqcup 1_{X^*} &= 1_{X^*} \sqcup w = w, \\ \forall x, y \in X, \forall u, v \in X^*, \quad xu \sqcup yv &= x(u \sqcup yv) + y(xu \sqcup v), \end{aligned} \quad (2)$$

or by their dual co-products,  $\Delta = \Delta_{\text{conc}}$  and  $\Delta = \Delta_{\sqcup}$ , defined by, for any  $w \in X^*$  by,

$$\begin{aligned} \Delta_{\text{conc}}(w) &= \sum_{w=uv} u \otimes v \\ \Delta_{\sqcup}(w) &= \sum_{I+J=[1..|w|]} w[I] \otimes w[J] \end{aligned} \quad (3)$$

One gets two Hopf algebras

$$\mathcal{H}_{\sqcup} = (\mathbb{Q}\langle X \rangle, \text{conc}, 1_{X^*}, \Delta_{\sqcup}, \epsilon, a_{\bullet}) \text{ and}$$

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<sup>2</sup>That is to say comes by dualization of a comultiplication.

<sup>3</sup>In the sequel, the order on the words will be understood as the lexicographic by length total ordering.

$$\mathcal{H}_{\sqcup}^\vee = (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*}, \Delta_{\text{conc}}, \epsilon, a_{\sqcup}) \quad (4)$$

mutually dual with respect to the pairing given by

$$(\forall u, v \in X^*)(\langle u \mid v \rangle = \delta_{u,v}) . \quad (5)$$

and with, for any  $x_{i_1}, \dots, x_{i_r} \in X$  and  $P \in \mathbb{Q}\langle X \rangle$ ,

$$\begin{aligned} \epsilon(P) &= \langle P \mid 1_{X^*} \rangle, \\ a_{\sqcup}(w) &= a_\bullet(w) = (-1)^r x_{i_r} \dots x_{i_1}, \end{aligned} \quad (6)$$

By the theorem of Cartier-Quillen-Milnor and Moore (CQMM in the sequel), the connected, graded positively, co-commutative Hopf algebra  $\mathcal{H}_{\sqcup}$  is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements which here is  $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$ . Hence, from any basis of the free algebra  $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$  one can<sup>4</sup> complete, by the Poincaré-Birkhoff-Witt theorem, a linear basis  $\{b_w\}_{w \in X^*}$  for  $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle) = \mathbb{Q}\langle X \rangle$  (see below (9) for an example of such a construction), and, when the basis is finely homogeneous, one can construct, by duality, a basis  $\{\check{b}_w\}_{w \in X^*}$  of  $\mathcal{H}_{\sqcup}$  (viewed as a  $\mathbb{Q}$ -module) such that :

$$\forall u, v \in X^*, \quad \langle \check{b}_u \mid b_v \rangle = \delta_{u,v} . \quad (7)$$

For  $w = l_1^{i_1} \dots l_k^{i_k}$  with  $l_1, \dots, l_k \in \mathcal{L}yn X$ ,  $l_1 > \dots > l_k$

$$\check{b}_w = \frac{\check{b}_{l_1}^{\sqcup i_1} \sqcup \dots \sqcup \check{b}_{l_k}^{\sqcup i_k}}{i_1! \dots i_k!} . \quad (8)$$

For example, Chen, Fox and Lyndon [5] constructed the PBW-Lyndon basis  $\{P_w\}_{w \in X^*}$  for  $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle)$  as follows

$$\begin{aligned} P_x &= x && \text{for } x \in X, \\ P_l &= [P_s, P_r] && \text{for } l \in \mathcal{L}yn X, \text{ standard factorization of } l = (s, r), \\ P_w &= P_{l_1}^{i_1} \dots P_{l_k}^{i_k} && \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k, l_1, \dots, l_k \in \mathcal{L}yn X. \end{aligned} \quad (9)$$

Schützenberger and his school constructed, the linear basis  $\{S_w\}_{w \in X^*}$  for  $\mathcal{A} = (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$  by duality (w.r.t. eq.5) and obtained the transcendence basis of  $\mathcal{A}$   $\{S_l\}_{l \in \mathcal{L}yn X}$  as follows<sup>5</sup>

$$S_l = x S_u, \quad \text{for } l = xu \in \mathcal{L}yn X, \quad (10)$$

$$S_w = \frac{S_{l_1}^{\sqcup i_1} \sqcup \dots \sqcup S_{l_k}^{\sqcup i_k}}{i_1! \dots i_k!} \quad \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 > \dots > l_k. \quad (11)$$

<sup>4</sup>The basis can be reindexed by Lyndon words and then one uses the canonical factorization of the words.

<sup>5</sup>Therefore  $\mathcal{A}$  is a polynomial algebra  $\mathcal{A} \simeq \mathbb{Q}[\mathcal{L}yn X]$ .

After that, Mélançon and Reutenauer [16] proved that<sup>6</sup>, for any  $w \in X^*$ ,

$$P_w = w + \sum_{v > w, |v|=|w|} c_v v \quad \text{and} \quad S_w = w + \sum_{v < w, |v|=|w|} c_v v. \quad (12)$$

On other words, the elements of the bases  $\{S_w\}_{w \in X^*}$  and  $\{P_w\}_{w \in X^*}$  are upper and lower triangular respectively and are multihomogeneous.

Moreover, thanks to the duality of the bases  $\{P_w\}_{w \in X^k}$  and  $\{S_w\}_{w \in X^k}$ , if  $\mathcal{D}_X$  denotes the diagonal series over  $X$  one has

$$\mathcal{D}_X = \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \text{Lyn} X}^{\searrow} \exp(S_l \otimes P_l). \quad (13)$$

In fact as stated in [16], this factorization holds in the framework of enveloping algebras and it will be shown in detail how to handle this framework even in the absence of any basis (it is CQMM analytic form **TODO explain**).

ACKNOWLEDGEMENTS. —

DEDICATION. —

### 3 General results on summability and duality

Let  $Y = \{y_i\}_{i \in I}$  be a totally ordered alphabet. The free monoid and the set of Lyndon words, over  $Y$ , are denoted respectively by  $Y^*$  and  $\text{Lyn} Y$ . The neutral of  $Y^*$  (and then of  $A(Y)$ ) is denoted by  $1_{Y^*}$ .

#### 3.1 Total algebras and duality

##### 3.1.1 Series and infinite sums

In the sequel, we will need to construct spaces of functions on different monoids (mainly direct products of free monoids). We set, once for all the general construction of the corresponding convolution algebra.

Let  $A$  be a unitary commutative ring and  $M$  a monoid. Let us denote  $A^M$  the set<sup>7</sup> of all (graphs of) mappings  $M \rightarrow A$ . This set is endowed with its classical structure of module. In order to extend the product defined in  $A[M]$  (the

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<sup>6</sup> Recall that the duality preserves the (multi)homogeneous degrees and interchanges the triangularity of polynomials [16]. For that, one can construct the triangular matrices  $M$  and  $N$  admitting as entries the coefficients of the multihomogeneous triangular polynomials,  $\{P_w\}_{w \in X^k}$  and  $\{S_w\}_{w \in X^k}$  in the basis  $\{w\}_{w \in X^*}$  respectively :

$$M_{u,v} = \langle P_u \mid v \rangle \quad \text{and} \quad N_{u,v} = \langle S_u \mid v \rangle.$$

The triangular matrices  $M$  and  $N$  are unipotent and satisfy the identity  $N = ({}^t M)^{-1}$ .

<sup>7</sup>In general  $Y^X$  is the set of all mappings  $X \rightarrow Y$  [2] Ch 2.5.2

algebra of the monoid  $M$ ), it is essential that, in the sums

$$f * g(m) = \sum_{m \in M} \sum_{uv=m} f(u)g(v) \quad (14)$$

the inner sum  $\sum_{uv=m} f(u)g(v)$  make sense. For that, we suppose that the monoid  $M$  fulfills condition “D” (be of finite decomposition type [3] Ch III.10). Formally, we say that  $M$  satisfies condition “D” iff, for all  $m \in M$ , the set

$$\{(u, v) \in M \times M \mid uv = m\} \quad (15)$$

is finite. In this case eq.14 endows  $A^M$  with the structure of a AAU. This algebra is traditionnaly called the total algebra of  $M$  (see [3] Ch III.10) and has very much to do with the series<sup>8</sup>. It will be, here (with a slight abuse of denotation which does not cause ambiguity) denoted  $A\langle\langle M \rangle\rangle$ .

The pairing

$$A\langle\langle M \rangle\rangle \otimes A[M] \longrightarrow A \quad (16)$$

defined by<sup>9</sup>

$$\langle f \mid g \rangle := \sum_{m \in M} f(m)g(m) \quad (17)$$

allows to see every element of the total algebra as a linear form on the module  $A[M]$ . One can check easily that, through this pairing, one has

$$A\langle\langle M \rangle\rangle \simeq (A[M])^* .$$

One says that a family  $(f_i)_{i \in I}$  of  $A\langle\langle M \rangle\rangle$  is summable [1] iff, for every  $m \in M$ , the mapping  $i \mapsto \langle f_i \mid m \rangle$  is finitely supported. In this case, the sum  $\sum_{i \in I} f_i$  is exactly the mapping  $m \mapsto \sum_{i \in I} \langle f_i \mid m \rangle$  so that, one has by definition

$$\langle \sum_{i \in I} f_i \mid m \rangle = \sum_{i \in I} \langle f_i \mid m \rangle . \quad (18)$$

To end with, let us remark that the set  $M_1 \otimes M_2 = \{u \otimes v\}_{(u,v) \in M_1 \times M_2}$  is a (monoidal) basis of  $A[M_1] \otimes A[M_2]$  and  $M_1 \otimes M_2$  is a monoid (in the product algebra  $A[M_1] \otimes A[M_2]$ ) isomorphic to the direct product  $M_1 \times M_2$ .

### 3.1.2 Summable families in Hom spaces.

In fact,  $A\langle\langle M \rangle\rangle \simeq (A[M])^* = \text{Hom}(A[M], A)$  and the notion of summability developed above can be seen as a particular case of that of a family of endomorphisms  $f_i \in \text{Hom}(V, W)$  for which  $\text{Hom}(V, W)$  appears as a complete space. It is indeed the pointwise convergence for the discrete topology. We will not detail these considerations here.

The definition is similar of that of a summable family of series [1], viewed as a family of linear forms.

<sup>8</sup>In fact, the algebra of commutative (resp. noncommutative) series on an alphabet  $X$  is the total algebra of the free commutative (resp.  $X^*$ ) monoid on  $X$

<sup>9</sup>Here  $A[M]$  is identified with the submodule of finitely supported functions  $M \rightarrow A$ .

**Definition 1.** *i) A family  $(f_i)_{i \in I}$  of elements in  $\text{Hom}(V, W)$  is said to be summable iff for all  $x \in V$ , the map  $i \mapsto f_i(x)$  has finite support. As a quantized criterium it reads*

$$(\forall x \in V)(\exists F \subset_{\text{finite}} I)(\forall i \notin F)(f_i(x) = 0) \quad (19)$$

*ii) If the family  $(f_i)_{i \in I} \in \text{Hom}(V, W)^I$  fulfils the condition 19 above its sum is given by*

$$\left(\sum_{i \in I} f_i\right)(x) = \sum_{i \in I} f_i(x) \quad (20)$$

It is an easy exercise to show that the mapping  $V \rightarrow W$  defined by the equation 20 is in fact in  $\text{Hom}(V, W)$ . Remark that, as the limiting process is defined by linear conditions, if a family  $(f_i)_{i \in I}$  is summable, so is

$$(a_i f_i)_{i \in I} \quad (21)$$

for an arbitrary family of coefficients  $(a_i)_{i \in I} \in A^I$ .

This tool will be used in section (3.2) to give an analytic presentation of the theorem of Cartier-Quillen-Milnor-Moore in the case when  $V = W = \mathcal{B}$  is a bialgebra.

The most interesting feature of this operation is the interversion of sums. Let us state it formally as a proposition the proof of which is left to the reader.

**Proposition 1.** *Let  $(f_i)_{i \in I}$  be a family of elements in  $\text{Hom}(V, W)$  and  $(I_j)_{j \in J}$  be a partition of  $I$  ([2] ch II §4 n° 7 Def. 6), then TFAE*

*i)  $(f_i)_{i \in I}$  is summable*

*ii) for all  $j \in J$ ,  $(f_i)_{i \in I_j}$  is summable and the family  $(\sum_{i \in I_j} f_i)_{j \in J}$  is summable. In these conditions, one has*

$$\sum_{i \in I} f_i = \sum_{j \in J} \left(\sum_{i \in I_j} f_i\right) \quad (22)$$

We derive at once from this the following practical criterium for double sums.

**Proposition 2.** *Let  $(f_{\alpha, \beta})_{(\alpha, \beta) \in A \times B}$  be a doubly indexed summable family in  $\text{Hom}(V, W)$ , then, for fixed  $\alpha$  (resp.  $\beta$ ) the “row-families”  $(f_{\alpha, \beta})_{\beta \in B}$  (resp. the “column-families”  $(f_{\alpha, \beta})_{\alpha \in A}$ ) are summable and their sums are summable. Moreover*

$$\sum_{(\alpha, \beta) \in A \times B} f_{\alpha, \beta} = \sum_{\alpha \in A} \sum_{\beta \in B} f_{\alpha, \beta} = \sum_{\beta \in B} \sum_{\alpha \in A} f_{\alpha, \beta} . \quad (23)$$

### 3.1.3 Substitutions

Let  $\mathcal{A}$  be a AAU and  $f \in \mathcal{A}$ . For every polynomial  $P \in A\langle X \rangle = A[X]$ , one can compute  $P(f)$  by

$$P(f) = \sum_{n \geq 0} \langle P \mid X^n \rangle f^n \quad (24)$$

one checks at once that  $P \mapsto P(f)$  is a morphism<sup>10</sup> of AAU between  $A[X]$  and  $\mathcal{A}$ . Moreover, this morphism is compatible with the substitutions as one checks easily that, for  $Q \in A[X]$

$$P(Q)(f) = P(Q(f)) \quad (25)$$

(it suffices to check that  $P \mapsto P(Q)(f)$  and  $P \mapsto P(Q(f))$  are two morphisms which coincide at  $P = X$ ).

In order to substitute within series, one needs some limiting process. The framework of  $\mathcal{A} = \text{Hom}(V, W)$  and summable families will be here sufficient (see paragraph 3.1.2). We suppose that  $(V, \delta_V, \epsilon_V)$  is a co-AAU and that  $(W, \mu_W, 1_W)$  is a AAU. Then  $(\text{Hom}(V, W), *, e)$  is a AAU (with  $e = 1_W \circ \epsilon_V$ ). A series  $S \in A[[X]]$  and  $f \in \text{Hom}(V, W)$  being given, we say that  $f \in \text{Dom}(S)$  iff the family  $(\langle S \mid X^n \rangle f^{*n})_{n \geq 0}$  is summable. It is easy to check that, if  $f \in \text{Dom}(S) \cap \text{Dom}(T)$  and  $\alpha \in A$ , one has

$$(\alpha S)(f) = \alpha S(f) ; (S + T)(f) = S(f) + T(f) \quad (26)$$

and

$$(TS)(f) = T(f) * S(f) . \quad (27)$$

If  $((f)^{*n})_{n \geq 0}$  is summable and  $S(0) = 0$  then

$$f \in \text{Dom}(S) \cap \text{Dom}(T(S)) ; S(f) \in \text{Dom}(T) \quad (28)$$

and

$$T(S)(f) = T(S(f)) \quad (29)$$

*Proof.* Let us first prove eq.27 . As  $f \in \text{Dom}(S) \cap \text{Dom}(T)$ , the families  $(\langle S \mid X^n \rangle f^{*n})_{n \geq 0}$  and  $(\langle T \mid X^m \rangle f^{*m})_{m \geq 0}$  are summable, then so is

$$\left( \langle T \mid X^m \rangle f^{*m} * \langle S \mid X^n \rangle f^{*n} \right)_{n, m \geq 0} \quad (30)$$

as, for every  $x \in V$ ,  $\delta(x) = \sum_{i=1}^N x_i^{(1)} \otimes x_i^{(2)}$  and for every  $i \in I$ ,

$$\text{supp}_{w.r.t. m}(\langle T \mid X^m \rangle f^{*m}(x_i^{(1)})) ; \text{supp}_{w.r.t. n}(\langle S \mid X^n \rangle f^{*n}(x_i^{(2)}))$$

are finite. Then outside of the cartesian product of the (finite) union of these supports, the product

$$(\langle T \mid X^m \rangle f^{*m} * \langle S \mid X^n \rangle f^{*n})(x) = \mu_W((\langle T \mid X^m \rangle f^{*m} \otimes \langle S \mid X^n \rangle f^{*n})(\delta(x))) \quad (31)$$

is zero. Hence the summability.

Now

$$T(f) * S(f) = \sum_{m=0}^{\infty} (\langle T \mid X^m \rangle f^{*m}) * \sum_{n=0}^{\infty} (\langle S \mid X^n \rangle f^{*n}) =$$

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<sup>10</sup>In case  $\mathcal{A}$  is a geometric space, this morphism is called “evaluation at  $f$ ” and corresponds to a Dirac measure.

$$\begin{aligned}
& \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\langle T \mid X^m \rangle \langle S \mid X^n \rangle f^{*n+m}) = \\
& \sum_{s=0}^{\infty} \left( \sum_{n+m=s}^{\infty} \langle T \mid X^m \rangle \langle S \mid X^n \rangle \right) f^{*s} = \\
& \sum_{s=0}^{\infty} (\langle TS \mid X^s \rangle) f^{*s} = (TS)(f)
\end{aligned} \tag{32}$$

We now prove the statements (28) and (29). If  $((f)^{*n})_{n \geq 0}$  is summable then  $f$  belongs to all domains (i.e. is universally substituable) by virtue of eq.21 . For all  $x \in V$ , it exists  $N_x \in \mathbb{N}$  such that

$$n > N_x \implies (f)^{*n}(x) = 0 .$$

Now, for  $S$  such that  $S(0) = 0$ , one has  $S = \sum_{n=1}^{\infty} \langle S \mid X^n \rangle X^n$  and then  $S^k = \sum_{n=k}^{\infty} \langle S^k \mid X^n \rangle X^n$ . Now, in view of eq.27 , one has

$$S(f)^{*n}(x) = S^n(f)(x) = \sum_{m=n}^{\infty} \langle S^n \mid X^m \rangle (f)^{*m}(x) \tag{33}$$

which is zero for  $n > N_x$ . Hence the summability of  $(S(f)^{*n})_{n \geq 0}$  which implies that  $S(f) \in \text{Dom}(T)$ . The family  $(\langle T \mid X^n \rangle \langle S^n \mid X^m \rangle (f)^{*m})_{(n,m) \in \mathbb{N}^2}$  is summable because, if  $x \in V$  and if  $n$  or  $m$  is greater than  $N_x$  then

$$\langle T \mid X^n \rangle \langle S^n \mid X^m \rangle (f)^{*m}(x) = 0 \tag{34}$$

thus  $T(S(f))$  is the sum



$$\begin{aligned}
T(S(f)) &= \sum_{n=0}^{\infty} \langle T \mid X^n \rangle S(f)^{*n} = \sum_{n=0}^{\infty} \langle T \mid X^n \rangle \sum_{m=n}^{\infty} \langle S^n \mid X^m \rangle (f)^{*m} = \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle T \mid X^n \rangle \langle S^n \mid X^m \rangle (f)^{*m} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} \langle T \mid X^n \rangle \langle S^n \mid X^m \rangle \right) (f)^{*m} = \\
&= \sum_{m=0}^{\infty} \langle T(S) \mid X^m \rangle (f)^{*m} = T(S)(f)
\end{aligned} \tag{35}$$

□

In the free case (i.e.  $V = W$  are the bialgebra  $(A\langle X \rangle, \text{conc}, 1_{X^*}, \Delta_{\sqcup}, \epsilon)$ ), one has a very useful representation of the convolution algebra  $\text{Hom}(V, W)$  through images of the diagonal series. This representation will provide us the key lemma (2). Let

$$\mathcal{D}_X = \sum_{w \in X^*} w \otimes w.$$

be the diagonal series attached to  $X$ .

**Proposition 3.** *Let  $A$  be a commutative unitary ring and  $X$  an alphabet. Then*

i) *For every  $f \in \text{End}(A\langle X \rangle)$ , the family  $(u \otimes f(u))_{u \in X^*}$  is summable in  $A\langle\langle X^* \otimes X^* \rangle\rangle$ .*

ii) *The representation*

$$f \mapsto \rho(f) = \sum_{u \in X^*} u \otimes f(u) \tag{36}$$

*is faithful from  $(\text{End}(A\langle X \rangle), *)$  to  $(A\langle\langle X^* \otimes X^* \rangle\rangle, \sqcup \otimes \text{conc})$ . In particular, for  $f \in \text{End}(A\langle X \rangle)$  and  $P \in A[X]$ , one has*

$$\rho(P(f)) = P(\rho(f)) \tag{37}$$

iii) *If  $f(1_{X^*}) = 0$  and  $S \in A[[X]]$  is a series, then  $(\rho(f)^n)_{n \geq 0}$  is summable in  $(A\langle\langle X^* \otimes X^* \rangle\rangle, \sqcup \otimes \text{conc})$  and*

$$\rho(S(f)) = S(\rho(f)) \tag{38}$$

*Proof.* (of Prop.(3)) Let us compute

$$\begin{aligned}
\rho(f)(\sqcup \otimes \text{conc})\rho(g) &= \sum_{u, v \in X^*} (u \otimes f(u)(\sqcup \otimes \text{conc})(v \otimes g(v))) = \\
&= \sum_{u, v \in X^*} (u \sqcup v \otimes (\text{conc}(f(u) \otimes g(v)))) =
\end{aligned}$$

$$\begin{aligned}
& \sum_{u,v \in X^*} \sum_{w \in X^*} (\langle u \sqcup v \mid w \rangle w \otimes \text{conc}(f(u)g(v))) = \\
& \sum_{w \in X^*} w \otimes \left( \sum_{u,v \in X^*} (\langle u \sqcup v \mid w \rangle \text{conc}(f(u)g(v))) \right) = \\
& \sum_{w \in X^*} w \otimes \left( \sum_{u,v \in X^*} (\langle u \otimes v \mid \Delta(w) \rangle \text{conc}(f(u)g(v))) \right) = \\
& \sum_{w \in X^*} w \otimes (\text{conc}(f \otimes g) \Delta(w)) = \sum_{w \in X^*} w \otimes (f * g(w)) \quad (39)
\end{aligned}$$

□

## 3.2 Theorem of Cartier-Quillen-Milnor-Moore (analytic form)

### 3.2.1 General properties of bialgebras

From now on, we suppose that  $A$  be a unitary commutative  $\mathbb{Q}$ -algebra (i.e.  $\mathbb{Q} \subset A$ ).

Let  $(\mathcal{B}, \mu, e_{\mathcal{B}}, \Delta, \epsilon)$  be a (general)  $A$ -bialgebra. One can always consider the Lie algebra of primitive elements  $\text{Prim}(\mathcal{B})$  and build the map  $j_{\mathcal{B}} : \mathcal{U}(\text{Prim}(\mathcal{B})) \rightarrow \mathcal{B}$ . Then,  $\mathcal{A} = j_{\mathcal{B}}(\mathcal{U}(\text{Prim}(\mathcal{B})))$  is the subalgebra generated by the primitive elements. It is not difficult to see that  $\mathcal{A}$  is a sub-bialgebra of  $\mathcal{B}$  as, for any list of primitive elements  $L = [g_1, g_2, \dots, g_n]$ , one has

$$\Delta(g_1 g_2 \dots g_n) = \Delta(L[\{1, 2, \dots, n\}]) = \sum_{I+J=\{1,2,\dots,n\}} L[I] \otimes L[J] \quad (40)$$

where, for  $I = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, n\}$ ,

$$L[I] = g_{i_1} g_{i_2} \dots g_{i_k} . \quad (41)$$

From (40) one gets also that  $j_{\mathcal{B}}$  is a morphism of bialgebras. In order to prove that it is always into, we need to construct the arrows  $\sigma, \tau$  which are a decomposition of a section of  $j_{\mathcal{B}}$ . Let us remark that, when  $\text{Prim}(\mathcal{B})$  is free as a

$$\begin{array}{ccccc}
\text{Prim}(\mathcal{B}) & \xrightarrow{i_{\mathcal{A},P}} & \mathcal{A} & \xrightarrow{i_{\mathcal{B},\mathcal{A}}} & \mathcal{B} \\
\downarrow i_{\mathcal{U},P} & \nearrow j_{\mathcal{B}} & \downarrow \sigma & & \\
\mathcal{U}(\mathcal{B}) & \xleftarrow{\tau} & T(\text{Prim}(\mathcal{B})) & & 
\end{array}$$

Figure 1: The sub-bialgebra  $\mathcal{A}$  generated by primitive elements.

$A$ -module, the proof of this fact is a consequence of the PBW theorem<sup>11</sup>. But,

<sup>11</sup>See [4] Ch2 §1 n° 6 th 1 for a field of characteristic zero and §1 Ex. 10 for the free case (over a ring  $A$  with  $\mathbb{Q} \subset A$ ).

here, we will construct the section in the general case using projectors which are now classical for the free case but which still can be computed analytically [16] as they lie in  $\mathbb{Q}[[X]]$  and still converge in  $\mathcal{A}$ .

*Proof.* (Injectivity of  $j_{\mathcal{B}}$ , construction of the section  $\tau \circ \sigma$ ). —

Let  $\mathcal{A}$  be the subalgebra of  $\mathcal{B}$  generated by  $\text{Prim}(\mathcal{B})$ , it is straightforward to check that  $\text{Im}(j_{\mathcal{B}}) = \mathcal{A}$ .

Remark that all series  $\sum_{n \geq 0} a_n (I_+)^{*n}$  are summable on  $\mathcal{A}$  (not in general on  $\mathcal{B}$  for example in case  $\mathcal{B}$  contains non-trivial group-like elements).

We define

$$c = \log_*(I) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (I_+)^{*n} \quad (42)$$

and remark that, in view of Prop. (3), in the case when  $\mathcal{B} = A\langle X \rangle$  one has  $\mathcal{A} = \mathcal{B}$  and, with  $S(X) = \log(1 + X)$

$$\begin{aligned} \sum_{w \in X^*} w \otimes \pi_1(w) &= \rho(\log(I)) = \rho(S(I^+)) = S(\rho(I^+)) = \\ S\left(\sum_{\substack{w \in X^* \\ w \neq 1_{X^*}}} w \otimes w\right) &= S(\mathcal{D}_X - 1_{X^*} \otimes 1_{X^*}) = \log(\mathcal{D}_X) \end{aligned} \quad (43)$$

We first prove that  $\pi_{1,\mathcal{A}}$  is a projector  $\mathcal{A} \rightarrow \text{Prim}(\mathcal{B})$ . The key point is that  $\Delta_{\mathcal{A}}$  (the restriction of the comultiplication to  $\mathcal{A}$ ) is a morphism of bialgebras<sup>12</sup>  $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ . We begin by proving that  $\Delta_{\mathcal{A}}$  “commutes” with the convolution. This is a consequence of the following property

**Lemma 1.** *i) Let  $f_i \in \text{End}(\mathcal{B}_i)$ , be such that  $\varphi f_1 = f_2 \varphi$ .*

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{\varphi} & \mathcal{B}_2 \\ f_1 \downarrow & & \downarrow f_2 \\ \mathcal{B}_1 & \xrightarrow{\varphi} & \mathcal{B}_2 \end{array}$$

Figure 2: Intertwining with a morphism of bialgebras (the functions of  $f_i$  below will be computed with the respective convolution products).

*i) Then, if  $P \in A[X]$ , one has*

$$\varphi P(f_1) = P(f_2) \varphi. \quad (44)$$

*ii) If the series  $\sum_{n \geq 0} (I_i^+)^{*n}$ ,  $i = 1, 2$  are summable, if  $f_1(1) = 0$  (which implies  $f_2(1) = 0$ ) and  $S \in A[[X]]$ , then the families  $(\langle S \mid X^n \rangle f_i^{*n})_{n \in \mathbb{N}}$  are summable, we denote  $S(f_i)$  their sums (this definition is coherent with the preceding when*

<sup>12</sup>In fact it is the case for any cocommutative bialgebra, be it generated by its primitive elements or not.

$S$  is a polynomial).

One has, for the convolution product,

$$\varphi S(f_1) = S(f_2)\varphi . \quad (45)$$

*Proof.* The only delicate part is (ii). First, one remarks that, if  $\varphi$  is a morphism of bialgebras, one has

$$(\varphi \otimes \varphi) \circ \Delta_1^+ = \Delta_2^+ \circ \varphi \quad (46)$$

then, the image by  $\varphi$  of an element of order less than  $N$  (i.e. such that  $\Delta_1^{+(N)}(x) = 0$ ) is of order less than  $N$ . Let now  $S$  be an univariate series  $S = \sum_{k=0}^{\infty} a_k X^k$ . For every element  $x$  of order less than  $N$  and  $f \in \text{End}(\mathcal{B})$  such that, one has

$$\begin{aligned} S(f)(x) &= \sum_{k=0}^{\infty} a_k f^{*k}(x) = \sum_{k=0}^{\infty} a_k \mu^{(k-1)} f^{\otimes k} \Delta^{(k-1)}(x) \\ &= \sum_{k=0}^{\infty} a_k \mu^{(k-1)} (f^{\otimes k}) \circ (I_+^{\otimes k}) \Delta^{(k-1)}(x) \\ &= \sum_{k=0}^N a_k \mu^{(k-1)} (f^{\otimes k}) \Delta_+^{(k-1)}(x) . \end{aligned} \quad (47)$$

This proves, in view of (i) that  $\varphi \circ S(f_1) = S(f_2) \circ \varphi$ .  $\square$

We reprove now that  $\pi_1$  is a projector[16]  $\mathcal{B} \rightarrow \text{Prim}(\mathcal{B})$  by means of the following lemma.

In case  $\mathcal{B}$  is cocommutative, the comultiplication  $\Delta$  is a morphism of bialgebras, so one has

$$\Delta \circ \log_*(I) = \log_*(I \otimes I) \circ \Delta \quad (48)$$

But

$$\begin{aligned} \log_*(I \otimes I) &= \log_*((I \otimes e) * (e \otimes I)) \\ &= \log_*(I \otimes e) + \log_*(e \otimes I) \\ &= \log_*(I) \otimes e + e \otimes \log_*(I) \end{aligned} \quad (49)$$

Then

$$\Delta(\log_*(I)) = (\log_*(I) \otimes e + e \otimes \log_*(I)) \circ \Delta \quad (50)$$

which implies that  $\log_*(I)(\mathcal{B}) \subset \text{Prim}(\mathcal{B})$ . To finish to prove that  $\pi_1$  is a projector onto  $\text{Prim}(\mathcal{B})$ , one has just to remark that, for  $x \in \text{Prim}(\mathcal{B})$  and  $n \geq 2$   $(\text{Id}^+)^{*n}(x) = 0$  then

$$\log_*(I)(x) = \text{Id}^+(x) = x . \quad (51)$$

$\square$

Now, we consider

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n \geq 0} \frac{1}{n!} \pi_{1, [\mathcal{A}]}^{*n} . \quad (52)$$

where  $\pi_{1, [\mathcal{A}]} = \log_*(I_{\mathcal{A}})$ .

Let us prove that the summands form an resolution of unity.

First, one defines  $\mathcal{A}_{[n]}$  as the linear span of the powers  $\{P^n\}_{P \in \text{Prim}(\mathcal{B})}$  or, equivalently of the symmetrized products

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} P_{\sigma(1)} P_{\sigma(2)} \cdots P_{\sigma(n)} . \quad (53)$$

It is obvious that  $\text{Im}(\pi_{1, [\mathcal{A}]}^{*n}) \subset \mathcal{A}_{[n]}$ . We remark that

$$\pi_{1, [\mathcal{A}]}^{*n} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1, [\mathcal{A}]}^{\otimes n} \Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1, [\mathcal{A}]}^{\otimes n} I_+^{\otimes n} \Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)} \pi_{1, [\mathcal{A}]}^{\otimes n} \Delta_+^{(n-1)} \quad (54)$$

as  $\pi_{1, [\mathcal{A}]} I_+ = \pi_{1, [\mathcal{A}]}$ . Now, let  $P \in \text{Prim}(\mathcal{A})$ . We compute  $\pi_{1, [\mathcal{A}]}^{*n}(P^m)$ . Indeed, if  $m < n$ , one has

$$\pi_{1, [\mathcal{A}]}^{*n}(P^m) = \mu_{\mathcal{B}}^{n-1} \Delta_+^{n-1}(P^m) = 0 . \quad (55)$$

If  $n = m$ , one has, from (40)

$$\Delta_+^{n-1}(P^n) = n! P^{\otimes n} \quad (56)$$

and hence  $\pi_{1, [\mathcal{A}]}^{*n}$  is the identity on  $\mathcal{A}_{[n]}$ . If  $m > n$ , the nullity of  $\pi_{1, [\mathcal{A}]}^{*n}(P^m)$  is a consequence of the following lemma.

**Lemma 2.** *Let  $\mathcal{B}$  be a bialgebra and  $P$  a primitive element of  $\mathcal{B}$ . Then*

*i) The series  $\log_*(I)$  is summable on each power  $P^m$*

*ii)  $\log_*(I)(P^m) = 0$  for  $m > 2$*

*Proof.* i) As  $\Delta_+^{*N}(P^m) = 0$  for  $N > m$ , one has  $I_+^{*N}(P^m) = 0$  for these values.

ii) Let  $a$  be a letter, the morphism of AAU  $\varphi_P : A[a] \rightarrow \mathcal{B}$ , defined by

$$\varphi_P(a) = P \quad (57)$$

is, in fact a morphism of bialgebras and one checks easily that One has just to check that  $\pi_{1, [A[a]]}(a^m) = 0$  for  $m > 2$  which is a consequence of the general equality (see eq.43 )

$$\sum_{w \in X^*} (w \otimes \pi_1(w)) = \log\left(\sum_{w \in X^*} w \otimes w\right) \quad (58)$$

because, for  $Y = \{a\}$  (and then  $A\langle X \rangle = A[a]$ ) one has

$$\begin{array}{ccc}
A[a] & \xrightarrow{\varphi_P} & \mathcal{B} \\
I_{A[a]}^+ \downarrow & & \downarrow I_{\mathcal{B}}^+ \\
A[a] & \xrightarrow{\varphi_P} & \mathcal{B}
\end{array}$$

Figure 3: Intertwining with one primitive element.

$$\begin{aligned}
\log\left(\sum_{w \in X^*} w \otimes w\right) &= \log\left(\sum_{n \geq 0} a^n \otimes a^n\right) = \\
\log\left(\sum_{n \geq 0} \frac{1}{n!} (a \otimes a)^{(\sqcup \otimes \text{conc})^n}\right) &= \log(\exp(a \otimes a)) = a \otimes a \quad (59)
\end{aligned}$$

this proves that  $\pi_{1,[A]}^{*n}(\mathcal{A}_{[m]}) = 0$  for  $m \neq n$  and hence the summands of the sum

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n \geq 0} \frac{1}{n!} \pi_{1,[A]}^{*n} . \quad (60)$$

are pairwise orthogonal projectors with  $\text{Im}(\pi_{1,[A]}^{*n}) = \mathcal{A}_{[n]}$  and then

$$\mathcal{A} = \oplus_{n \geq 0} \mathcal{A}_{[n]} . \quad (61)$$

This decomposition permits to construct  $\sigma$  by

$$\sigma(P^n) = \frac{1}{n!} \Delta_+^{(n-1)}(P^n) \in T_n(\text{Prim}(\mathcal{B})) \quad (62)$$

for  $n \geq 1$  and, one sets  $\sigma(1_{\mathcal{B}}) = 1_{T(\text{Prim}(\mathcal{B}))}$ .

It is easy to check that  $j_{\mathcal{B}} \circ \tau \circ \sigma = \text{Id}_{\mathcal{A}}$  as  $\mathcal{A}$  is (linearly) generated by the powers  $(P^n)_{P \in \text{Prim}(\mathcal{B}), n \geq 0}$ .  $\square$

### 3.2.2 The theorem from the point of view of summability

The bialgebra  $\mathcal{B}$  being supposed cocommutative, we discuss the equivalent conditions under which we are in the presence of an enveloping algebra i.e.

$$\mathcal{B} \cong_{A\text{-bialg}} \mathcal{U}(\text{Prim}(\mathcal{B})) \quad (63)$$

from the point of view of the convergence of the series  $\log_*(I)^{13}$ . These conditions are known as the theorem of Cartier-Quillen-Milnor-Moore (CQMM).

<sup>13</sup>In a  $A$ -bialgebra, one can always consider the series of endomorphisms

$$\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (I^+)^{*n} . \quad (64)$$

The family  $(\frac{(-1)^{n-1}}{n} (I^+)^{*n})_{n \geq 0}$  is summable iff  $((I^+)^{*n})_{n \geq 0}$  is (use eq.21).

**Theorem 1.** [4] Let  $\mathcal{B}$  be a  $A$ -cocommutative bialgebra ( $A$  is a  $\mathbb{Q}$ -AAU) and  $\mathcal{A}$ , as above, the subalgebra generated by  $\text{Prim}(\mathcal{B})$ . Then, the following conditions are equivalent :

i)  $\mathcal{B}$  admits an increasing filtration

$$\mathcal{B}_0 = A.1_{\mathcal{B}} \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \cdots$$

compatible with the structures of algebra (i.e. for all  $p, q \in \mathbb{N}$ , one has  $\mathcal{B}_p \mathcal{B}_q \subset \mathcal{B}_{p+q}$ ) and coalgebra :

$$\forall n \in \mathbb{N}, \quad \Delta(\mathcal{B}_n) \subset \sum_{p+q=n} \mathcal{B}_p \otimes \mathcal{B}_q.$$

ii)  $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$  is summable in  $\text{End}(\mathcal{B})$ .

iii)  $\mathcal{B} = \mathcal{A}$ .

*Proof.* We prove

$$(ii) \implies (iii) \implies (i) \implies (ii) \quad (65)$$

(ii)  $\implies$  (iii). —

The image of  $j_{\mathcal{B}}$  is the subalgebra generated by the primitive elements. Let us prove that, when  $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$  is summable, one has  $\text{Im}(j_{\mathcal{B}}) = \mathcal{B}$ . The series  $\log(1 + X)$  is without constant term so, in virtue of (29) and the summability of  $((\text{Id}^+)^{*n})_{n \in \mathbb{N}}$ , one has

$$\exp(\log(e + \text{Id}^+)) = \exp(\log(1 + X))(\text{Id}^+) = 1_{\text{End}(\mathcal{B})} + \text{Id}^+ = e + \text{Id}^+ = I \quad (66)$$

Set  $\pi_1 = \log(e + \text{Id}^+)$ .

To end this part, let us compute, for  $x \in \mathcal{B}$

$$x = \exp(\pi_1)(x) = \left( \sum_{n \geq 0} \frac{1}{n!} \pi_1^{*n} \right)(x) = \left( \sum_{n=0}^N \frac{1}{n!} \mu^{(n-1)} \pi_1^{\otimes n} \right) \Delta^{(n-1)}(x) \quad (67)$$

where  $N$  is the first order for which  $\Delta^{+(n-1)}(x) = 0$  (as  $\pi_1 \circ \text{Id}^+ = \pi_1$ ). This proves that  $\mathcal{B}$  is generated by its primitive elements.

(iii)  $\implies$  (i). —

□

**Remark 1.** i) The equivalence (i)  $\iff$  (iii) is the classical CQMM theorem (see [4]). The equivalence with (ii) could be called the “Convolutional CQMM theorem”. The combinatorial aspects of this last one will be the subject of a forthcoming paper [CT, HNM, GHED Nguyen ?]

ii) When  $\text{Prim}(\mathcal{B})$  is free, we have  $\mathcal{B} \cong_{k\text{-bialg}} \mathcal{U}(\text{Prim}(\mathcal{B}))$  and  $\mathcal{B}$  is an enveloping algebra.

iii) The (counter) example is the following with  $A = k[x]$  ( $k$  is a field of characteristic zero). Let  $Y$  be an alphabet and  $A\langle Y \rangle$  be the usual free algebra (the space of non-commutative polynomials over  $Y$ ) and  $\epsilon$ , the “constant term” linear form. Let  $\text{conc}$  be the concatenation and  $\Delta$  the unshuffling. Then the bialgebra  $(A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta, \epsilon)$  is a Hopf algebra (it is the enveloping algebra of the Lie polynomials). Let  $A_+\langle Y \rangle = \ker(\epsilon)$  and, for  $N \geq 2$   $J_N = x^N \cdot A_+\langle Y \rangle$  then,  $J_N$  is a Hopf ideal and  $\text{Prim}(A\langle Y \rangle / (J_N))$  is never free (no basis).

## 4 Case study : $\phi$ -deformed stuffle

### 4.1 Results for the $\phi$ -deformed stuffle

Let  $Y = \{y_i\}_{i \in I}$  be still a totally ordered alphabet and  $A\langle Y \rangle$  be equipped with the  $\phi$ -deformed stuffle defined by

- i) for any  $w \in Y^*$ ,  $1_{Y^*} \boxtimes_{\phi} w = w \boxtimes_{\phi} 1_{Y^*} = w$ ,
- ii) for any  $y_i, y_j \in Y$  and  $u, v \in Y^*$ ,

$$y_i u \boxtimes_{\phi} y_j v = y_j (y_i u \boxtimes_{\phi} v) + y_i (u \boxtimes_{\phi} y_j v) + \phi(y_i, y_j) u \boxtimes_{\phi} v, \quad (68)$$

where  $\phi$  is an arbitrary mapping

$$\phi : Y \times Y \longrightarrow AY.$$

**Definition 2.** Let

$$\phi : Y \times Y \longrightarrow AY$$

defined by its structure constants

$$(y_i, y_j) \longmapsto \phi(y_i, y_j) = \sum_{k \in I} \gamma_{i,j}^k y_k.$$

**Proposition 4.** The recursion (68) defines a unique mapping

$$\boxtimes_{\phi} : Y^* \times Y^* \longrightarrow A\langle Y \rangle.$$

*Proof.* Let us denote  $(Y^* \times Y^*)_{\leq n}$  the set of words  $(u, v) \in Y^* \times Y^*$  such that  $|u| + |v| \leq n$ . We construct a sequence of mappings

$$\boxtimes_{\phi \leq n} : (Y^* \times Y^*)_{\leq n} \longrightarrow AY.$$

which satisfy the recursion of eq.68. For  $n = 0$ , we have only a preimage and  $\boxtimes_{\phi \leq 0}(1_{Y^*}) = 1_{Y^*} \otimes 1_{Y^*}$ . Suppose  $\boxtimes_{\phi \leq n}$  constructed and let  $(u, v) \in (Y^* \times Y^*)_{\leq n+1} \setminus (Y^* \times Y^*)_{\leq n}$ , i.e.  $|u| + |v| = n + 1$ . One has three cases :  $u = 1_{Y^*}$ ,  $v = 1_{Y^*}$  and  $(u, v) \in Y^+ \times Y^+$ . For the two first, one uses the initialisation of the recursion thus

$$\boxtimes_{\phi \leq n+1}(w, 1_{Y^*}) = \boxtimes_{\phi \leq n+1}(1_{Y^*}, w) = w$$



for the last case, write  $u = y_i u'$ ,  $v = y_j v'$  and use, to get

$$\mathfrak{U}_{\phi \leq n+1}(y_i u', y_j v') = y_i \mathfrak{U}_{\phi \leq n}(u', y_j v') + y_j \mathfrak{U}_{\phi \leq n}(y_i u', v') + y_{i+j} \mathfrak{U}_{\phi \leq n}(u', v')$$

this proves the existence of the sequence  $(\mathfrak{U}_{\phi \leq n})_{n \geq 0}$ . Every  $\mathfrak{U}_{\phi \leq n+1}$  extends the preceding so there is a mapping

$$\mathfrak{U}_{\phi} : Y^* \times Y^* \longrightarrow A\langle Y \rangle.$$

which extends all the  $\mathfrak{U}_{\phi \leq n+1}$  (the graph of which is the union of the graphs of the  $\mathfrak{U}_{\phi \leq n}$ ). This proves the existence. For unicity, just remark that, if there were two mappings  $\mathfrak{U}_{\phi}$ ,  $\mathfrak{U}'_{\phi}$ , the fact that they must fulfill the recursion (68) implies that  $\mathfrak{U}_{\phi} = \mathfrak{U}'_{\phi}$ .  $\square$

We still denote  $\phi$  and  $\mathfrak{U}_{\phi}$  the linear extension of  $\phi$  and  $\mathfrak{U}_{\phi}$  to  $AY \otimes AY$  and  $A\langle Y \rangle \otimes A\langle Y \rangle$  respectively.

Then  $\mathfrak{U}_{\phi}$  is a law of algebra (with  $1_{Y^*}$  as unit) on  $A\langle Y \rangle$ .

**Lemma 3.** *Let  $\Delta$  be the morphism  $A\langle Y \rangle \rightarrow A\langle\langle Y^* \otimes Y^* \rangle\rangle$  defined on the letters by*

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{n,m \in I} \gamma_{n,m}^s y_n \otimes y_m \quad (69)$$

*Then*

*i) for all  $w \in Y^+$  we have*

$$\Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v \in Y^+} \langle \Delta(w) \mid u \otimes v \rangle u \otimes v \quad (70)$$

*ii) for all  $u, v, w \in Y^*$ , one has*

$$\langle u \mathfrak{U}_{\phi} v \mid w \rangle = \langle u \otimes v \mid \Delta(w) \rangle^{\otimes 2} \quad (71)$$

*Proof.* i) By recurrence on  $|w|$ . If  $w = y_s$  is of length one, it is obvious from the definition. If  $w = y_s w'$ , we have, from the fact that  $\Delta$  is a morphism

$$\begin{aligned} \Delta(w) &= \left( y_s \otimes 1 + 1 \otimes w + \sum_{i,j \in I} \gamma_{i,j}^s y_i \otimes y_j \right) \\ &\quad \left( w' \otimes 1 + 1 \otimes w' + \sum_{u,v \in Y^+} \langle u \otimes v \mid \Delta(w') \rangle \right) \end{aligned} \quad (72)$$

the development of which proves that  $\Delta(w)$  is of the desired form.

ii) Let  $S(u, v) := \sum_{w \in Y^*} \langle u \otimes v \mid \Delta(w) \rangle w$ . It is easy to check (and left to the reader) that, for all  $u \in Y^*$ ,  $S(u, 1) = S(1, u) = u$ . Let us now prove that, for all  $y_i, y_j \in Y$  and  $u, v \in Y^*$

$$S(y_i u, y_j v) = y_i S(u, y_j v) + y_j S(y_i u, v) + \phi(y_i, y_j) S(u, v) \quad (73)$$

Indeed, remarking that  $\Delta(1) = 1 \otimes 1$ , one has

$$\begin{aligned}
S(y_i u, y_j v) &= \sum_{w \in Y^*} \langle y_i u \otimes y_j v \mid \Delta(w) \rangle w = \sum_{w \in Y^+} \langle y_i u \otimes y_j v \mid \Delta(w) \rangle w \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \Delta(y_s w') \rangle y_s w' \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \left( y_s \otimes 1 + 1 \otimes y_s + \sum_{n, m \in I} \gamma_{n, m}^s y_n \otimes y_m \right) \Delta(w') \rangle y_s w' \\
&= \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid (y_s \otimes 1) \Delta(w') \rangle y_s w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid (1 \otimes y_s) \Delta(w') \rangle y_s w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle y_i u \otimes y_j v \mid \left( \sum_{n, m \in I} \gamma_{n, m}^s y_n \otimes y_m \right) \Delta(w') \rangle y_s w' \\
&= \sum_{w' \in Y^*} \langle u \otimes y_j v \mid \Delta(w') \rangle y_i w' + \sum_{w' \in Y^*} \langle y_i u \otimes v \mid \Delta(w') \rangle y_j w' \\
&\quad + \sum_{y_s \in Y, w' \in Y^*} \langle u \otimes v \mid \gamma_{i, j}^s \Delta(w') \rangle y_s w' \\
&= y_i \sum_{w' \in Y^*} \langle u \otimes y_j v \mid \Delta(w') \rangle w' + y_j \sum_{w' \in Y^*} \langle y_i u \otimes v \mid \Delta(w') \rangle w' \\
&\quad + \sum_{y_s \in Y} \gamma_{i, j}^s y_s \sum_{w' \in Y^*} \langle u \otimes v \mid \Delta(w') \rangle w' \\
&= y_i S(u, y_j v) + y_j S(y_i u, v) + \phi(y_i, y_j) S(u, v)
\end{aligned}$$

then the computation of  $S$  shows that, for all  $u, v \in Y^*$ ,  $S(u, v) = u \bowtie_\phi v$  as  $S$  is bilinear, one has  $S = \bowtie_\phi$ .  $\square$

**Theorem 2.** *i) The law  $\bowtie_\phi$  is commutative if and only if the extension*

$$\phi : AY \otimes AY \longrightarrow AY$$

*is so.*

*ii) The law  $\bowtie_\phi$  is associative if and only if the extension*

$$\phi : AY \otimes AY \longrightarrow AY$$

*is so.*

*iii) Let  $\gamma_{x, y}^z := \langle \phi(x, y) \mid z \rangle$  be the structure constants of  $\phi$  (w.r.t. the basis  $Y$ ), then  $\bowtie_\phi$  is dualizable if and only if  $(\gamma_{x, y}^z)_{x, y, z \in X}$  is of finite decomposition*

type<sup>14</sup> in its superscript in the following sense

$$(\forall z \in X)(\#\{(x, y) \in X^2 \mid \gamma_{x,y}^z \neq 0\} < +\infty). \quad (74)$$

*Proof.* (i) First, let us suppose that  $\phi$  be commutative and consider  $T$ , the twist, i.e. the operator in  $A\langle\langle Y^* \otimes Y^* \rangle\rangle$  defined by

$$\langle T(S) \mid u \otimes v \rangle = \langle S \mid v \otimes u \rangle \quad (75)$$

it is left to the reader to prove that  $T$  is a morphism of algebras. If  $\phi$  is commutative, then so is the following diagram.

$$\begin{array}{ccc} Y & \xrightarrow{\Delta_{\boxtimes \phi}} & A\langle\langle Y^* \otimes Y^* \rangle\rangle \\ & \searrow \Delta_{\boxtimes \phi} & \downarrow T \\ & & A\langle\langle Y^* \otimes Y^* \rangle\rangle \end{array}$$

and, then, the two morphisms  $\Delta_{\boxtimes \phi}$  and  $T \circ \Delta_{\boxtimes \phi}$  coincide on the generators  $Y$  of the algebra  $A\langle Y \rangle$  and hence over  $A\langle Y \rangle$  itself. Now for all  $u, v, w \in Y^*$ , one has

$$\begin{aligned} \langle v \boxtimes_{\phi} u \mid w \rangle &= \langle v \otimes u \mid \Delta_{\boxtimes \phi}(w) \rangle = \langle u \otimes v \mid T \circ \Delta_{\boxtimes \phi}(w) \rangle = \\ &= \langle u \otimes v \mid \Delta_{\boxtimes \phi}(w) \rangle = \langle u \boxtimes_{\phi} v \mid w \rangle \end{aligned} \quad (76)$$

which proves that  $v \boxtimes_{\phi} u = u \boxtimes_{\phi} v$ . Conversely, if  $\boxtimes_{\phi}$  is commutative, one has, for  $i, j \in I$

$$\phi(y_j, y_i) = y_j \boxtimes_{\phi} y_i - (y_j \sqcup y_i) = y_i \boxtimes_{\phi} y_j - (y_i \sqcup y_j) = \phi(y_i, y_j) \quad (77)$$

(ii) Likewise, if  $\phi$  is associative, let us define the operators

$$\overline{\Delta_{\boxtimes \phi} \otimes I} : A\langle\langle Y^* \otimes Y^* \rangle\rangle \rightarrow A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle \quad (78)$$

by

$$\langle \overline{\Delta_{\boxtimes \phi} \otimes I}(S) \mid u \otimes v \otimes w \rangle = \langle S \mid (u \boxtimes_{\phi} v) \otimes w \rangle \quad (79)$$

and, similarly,

$$\overline{I \otimes \Delta_{\boxtimes \phi}} : A\langle\langle Y^* \otimes Y^* \rangle\rangle \rightarrow A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle \quad (80)$$

by

$$\langle \overline{I \otimes \Delta_{\boxtimes \phi}}(S) \mid u \otimes v \otimes w \rangle = \langle S \mid u \otimes (v \boxtimes_{\phi} w) \rangle \quad (81)$$

it is easy to check by direct calculation that they are well defined morphisms and that the following diagram

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<sup>14</sup>One can prove that, in case  $Y$  is a semigroup, the associated  $\phi$  is fulfills eq.74 iff  $Y$  fulfills “condition D” of Bourbaki (see [3])

$$\begin{array}{ccc}
Y & \xrightarrow{\Delta_{\boxplus_\phi}} & A\langle\langle Y^* \otimes Y^* \rangle\rangle \\
\Delta_{\boxplus_\phi} \downarrow & & \downarrow \overline{I \otimes \Delta_{\boxplus_\phi}} \\
A\langle\langle Y^* \otimes Y^* \rangle\rangle & \xrightarrow{\overline{\Delta_{\boxplus_\phi} \otimes I}} & A\langle\langle Y^* \otimes Y^* \otimes Y^* \rangle\rangle
\end{array}$$

is commutative. This proves that the two composite morphisms

$$\overline{\Delta_{\boxplus_\phi} \otimes I} \circ \Delta_{\boxplus_\phi}$$

and

$$\overline{I \otimes \Delta_{\boxplus_\phi}} \circ \Delta_{\boxplus_\phi}$$

coincide on  $Y$  and then on  $A\langle Y \rangle$ . Now, for  $u, v, w, t \in Y^*$ , one has

$$\begin{aligned}
\langle (u \boxplus_\phi v) \boxplus_\phi w \mid t \rangle &= \langle (u \boxplus_\phi v) \otimes w \mid \Delta_{\boxplus_\phi}(t) \rangle = \langle u \otimes v \otimes w \mid (\overline{\Delta_{\boxplus_\phi} \otimes I}) \Delta_{\boxplus_\phi}(t) \rangle = \\
&= \langle u \otimes v \otimes w \mid (\overline{I \otimes \Delta_{\boxplus_\phi}}) \Delta_{\boxplus_\phi}(t) \rangle = \langle u \otimes (v \boxplus_\phi w) \mid \Delta_{\boxplus_\phi}(t) \rangle = \langle u \boxplus_\phi (v \boxplus_\phi w) \mid t \rangle
\end{aligned}$$

which proves the associativity of the law  $\boxplus_\phi$ . Conversely, if  $\boxplus_\phi$  is associative, the direct expansion of the right hand side of

$$0 = (y_i \boxplus_\phi y_j) \boxplus_\phi y_k - y_i \boxplus_\phi (y_j \boxplus_\phi y_k) \quad (82)$$

proves the associativity of  $\phi$ .

iii) We suppose that  $(\gamma_{x,y}^z)_{x,y,z \in X}$  is of finite decomposition type in its superscript, in this case  $\Delta_{\boxplus_\phi}$  takes its values in  $A\langle Y \rangle \otimes A\langle Y \rangle$  therefore its dual, the law  $\boxplus_\phi$  is dualizable. Conversely, if  $\text{Im}(\Delta_{\boxplus_\phi}) \subset A\langle Y \rangle \otimes A\langle Y \rangle$ , one has, for every  $s \in I$

$$\sum_{n,m \in I} \gamma_{n,m}^s y_n \otimes y_m = \Delta(y_s) - (y_s \otimes 1 + 1 \otimes y_s) \in A\langle Y \rangle \otimes A\langle Y \rangle$$

which proves the claim.  $\square$

**From now on, we suppose that  $\phi : AY \otimes AY \rightarrow AY$  be an associative and commutative law (of algebra) on  $AY$ .**

**Theorem 3.** *Let  $A$  be a commutative ring with unit. Then if  $\phi$  is dualizable<sup>15</sup>, let  $\Delta_{\boxplus_\phi} : A\langle Y \rangle \rightarrow A\langle Y \rangle \otimes A\langle Y \rangle$  denote its dual comultiplication, then*

a)  $\mathcal{B}_\phi = (A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\boxplus_\phi}, \varepsilon)$  is a bialgebra.

---

<sup>15</sup>For the pairing defined by

$$\forall x, y \in Y, \quad \langle x \mid y \rangle = \delta_{x,y}$$

b) If  $A$  is a field of characteristic 0 then  $\mathcal{B}_\phi$  is an enveloping bialgebra if and only if the algebra  $AX$  admits an increasing filtration  $((AY)_n)_{n \in \mathbb{N}}$  with  $(AY)_0 = \{0\}$  and compatible with both the multiplication and the comultiplication  $\Delta_{\mathfrak{L}_\phi}$  i.e.

$$\begin{aligned} (AY)_p(AY)_q &\subset (AY)_{p+q} \\ \Delta_{\mathfrak{L}_\phi}((AY)_n) &\subset \sum_{p+q=n} (AY)_p \otimes (AY)_q . \end{aligned}$$

*Proof.* i) All the properties of bialgebra have been checked for

$$\mathcal{B}_\phi = (A\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\mathfrak{L}_\phi}, \varepsilon)$$

save one : the fact that  $\Delta_{\mathfrak{L}_\phi}$  be a morphism for the product. This is a consequence of the fact that, in the general case,

$$\Delta_{\mathfrak{L}_\phi} : A\langle Y \rangle \rightarrow A\langle Y^* \otimes Y^* \rangle$$

is a morphism of algebras.

ii) Let us suppose first that  $\mathcal{B}_\phi = \mathcal{U}G$  is a enveloping algebra. Then, the intersection of the standard increasing filtration with  $AY$  i.e.

$$(AY)_n := \text{span}(G^n) \cap AY$$

is compatible with product and coproduct and  $(AY)_0 := K.1_{\mathcal{U}G} \cap AY = \{0\}$ . Conversely let  $((AY)_n)_{n \in \mathbb{N}}$  be an increasing filtration of  $AY$  which fulfils the conditions of the theorem and set

$$(\mathcal{B}_\phi)_n = k.1_{\mathcal{B}_\phi} + \sum_{k \geq 0} \sum_{\substack{p_1 + p_2 + \dots + p_k = n \\ p_i > 0}} \text{span}((AY)_{p_1}(AY)_{p_2} \dots (AY)_{p_k}) \quad (83)$$

has the properties required for the application of the theorem of Cartier-Milnor-Moore. Hence  $\mathcal{B}_\phi$  is an enveloping algebra.  $\square$

In view of section (3.2), the antipode is computed by

$$a_{\mathfrak{L}_\phi} = (I)^{* - 1} = (e + I^+)^{* - 1} = \sum_{n \geq 0} (-1)^n (I^+)^{*n} \quad (84)$$

**TODO Dveloppeur la rcurson** With the co-unit

$$\forall P \in A\langle Y \rangle, \quad \epsilon(P) = \langle P \mid 1_{Y^*} \rangle, \quad (85)$$

and the antipode defined by, for any  $w = x_{i_1} \dots x_{i_r} \in Y^*$ ,

$$a_{\mathfrak{L}_\phi}(y_{i_1} \dots y_{i_r}) = - \sum_{k=1}^{r-1} a_{\mathfrak{L}_\phi}(y_{i_1} \dots y_{i_k}) \mathfrak{L}_\phi y_{i_{k+1}} \dots y_{i_r}, \quad (86)$$

one gets mutually dual Hopf algebras  $\mathcal{H}_{\mathfrak{L}_\phi} = (\mathbb{Q}\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\mathfrak{L}_\phi}, \epsilon, a_{\mathfrak{L}_\phi})$  and  $\mathcal{H}_{\mathfrak{L}_\phi}^\vee = (\mathbb{Q}\langle Y \rangle, \mathfrak{L}_\phi, 1_{Y^*}, \Delta_{\text{conc}}, \epsilon, a_{\mathfrak{L}_\phi})$ .

## 5 Conclusion

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